Geometric-Algebra Adaptive Filters

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Doctoral Dissertation Defense

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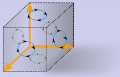

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Introduction

- Linear Algebra (LA) has been the mathematical *lingua franca* in signal processing.
- LA is a reliable tool for the derivation of regular adaptive filters.

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$$\min_{w} \left\| d(i) - u_i^T w_i \right\|^2 \tag{1}$$

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- Geometric Algebra
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  - Geometric product
  - Geometric calculus
- How can one use it for the benefit of AFs?

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Contributions

1. *Recast of central concepts of linear estimation into GA framework*
2. *GAAFs (standard shape)*
   - Design of GA Least-Mean Squares (GA-LMS);
   - Steady-state mean-square analysis.
3. *GAAFs (pose estimation)*
   - Design of GA-LMS for pose estimation;
   - Evaluation of the computational complexity;
   - Calculation of step-size bounds as a function of the PCDs points and their greatest dimension.
4. *Computational implementation* - openga.org

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The System Identification Problem

- The goal is to estimate the entries (coefficients) of an unknown plant (system) modeled by an $M \times 1$ vector w^o

$$d(i) = u_i^H w^o + v(i), \qquad (2)$$

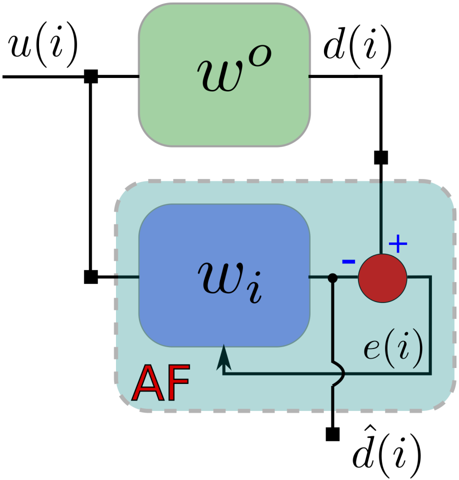

Figure 1 : The system identification scenario.

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Registration of Point Clouds

- 3D Point Clouds: Target (Red) and Source (Blue).
- No initial alignment.
- Typical problem in computer vision, particularly visual navigation (robots, drones, autonomous vehicles etc).

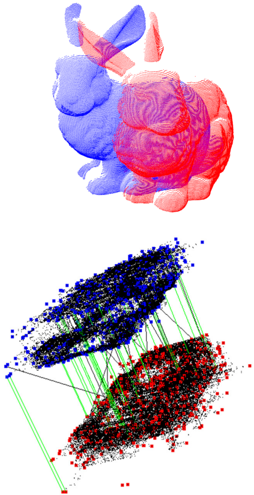
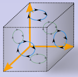


Figure 2 : Examples of Point Clouds.



Registration of Point Clouds

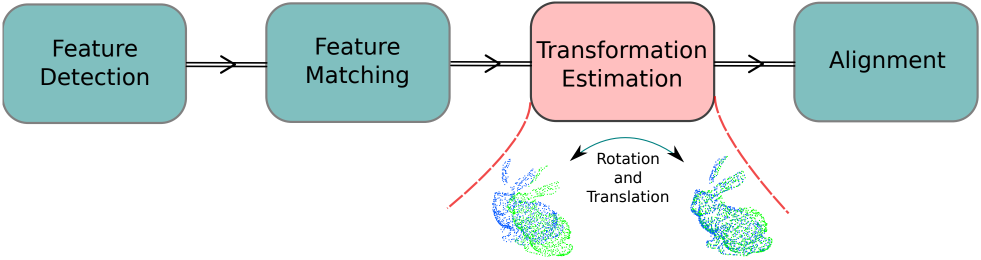


Figure 3 : Registration Pipeline.

- The goal is to match two PCDs (in this case, bunnies) which are initially unaligned. This work focus on the "Transformation Estimation" phase, where a new estimator based on GA and AFs is introduced.

Fundamentals of Geometric Algebra

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Elements of Geometric Algebra

The GA $\mathcal{G}(\mathbb{R}^n)$ is a geometric extension of $\mathbb{R}^n$ to represent orientation and magnitude [1].

- Vectors in $\mathbb{R}^n$ are also vectors in $\mathcal{G}(\mathbb{R}^n)$;
- *geometric product (GP):*

  Vectors a and b in $\mathbb{R}^n \rightarrow ab \triangleq a \cdot b + a \wedge b$;
- Noncommutative: $ab \neq ba$; Associative: $abc = (ab)c = a(bc)$;
- $a \cdot b = 0 \Rightarrow ab = a \wedge b = -b \wedge a = -ba$.

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Elements of Geometric Algebra

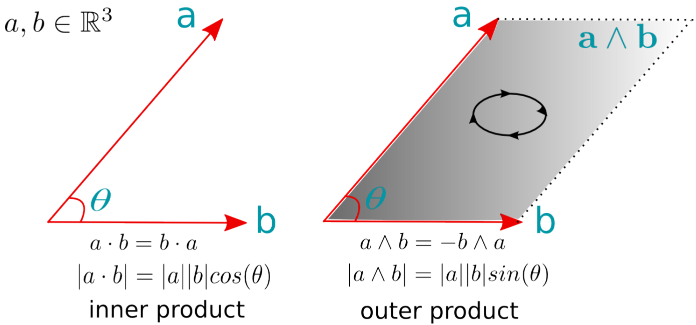

Figure 4 : Inner and outer products in $\mathbb{R}^3$. In the outer product case, the orientation of the circle defines the orientation of the area (bivector).

- From now on all products are Geometric Products (GP)

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Geometric Algebra of a Vector Space

In $\mathbb{R}^n$: (orthonormal basis $+\ 1$) $\xrightarrow{\text{GP}} 2^n$ elements $\in \mathcal{G}(\mathbb{R}^n)$

In $\mathbb{R}^3$: (orthonormal basis $+\ 1$) $\xrightarrow{\text{GP}} 2^3{=}8$ elements $\in \mathcal{G}(\mathbb{R}^3)$

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$$\{\gamma_1, \gamma_2, \gamma_3\} + \{1\}$$

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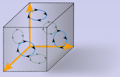
$\{\gamma_1, \gamma_2, \gamma_3\} + \{1\} \xrightarrow{\text{GP}}$



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$$\{\gamma_1, \gamma_2, \gamma_3\} + \{1\} \xrightarrow{\text{GP}} \{1, \underbrace{\gamma_1, \gamma_2, \gamma_3}_{\text{vectors}}, \underbrace{\gamma_{12}, \gamma_{23}, \gamma_{31}}_{\text{bivectors}}, \underbrace{\mathcal{I}}_{\text{trivector}}\}$$

$$\gamma_{ij} \triangleq \gamma_i \gamma_j, \quad \mathcal{I} \triangleq \gamma_1 \gamma_2 \gamma_3$$

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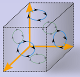
$\gamma_{ij} \triangleq \gamma_i \gamma_j, \quad \mathcal{I} \triangleq \gamma_1 \gamma_2 \gamma_3$

Multivector $A \in \mathcal{G}(\mathbb{R}^n) \rightarrow$ fundamental information block:

$$A = \underbrace{\langle A \rangle_0}_{\text{scalar}} + \underbrace{\langle A \rangle_1}_{\text{vector}} + \underbrace{\langle A \rangle_2}_{\text{bivector}} + \cdots = \sum_g \langle A \rangle_g. \tag{3}$$



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$$\boxed{\text{GA theory enables us to sum apples and oranges in a well-defined fashion!}}$$

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Geometric Algebra of a Vector Space

- $\mathcal{G}(\mathbb{R}^3)$: Complete Geometric Algebra of $\mathbb{R}^3$.

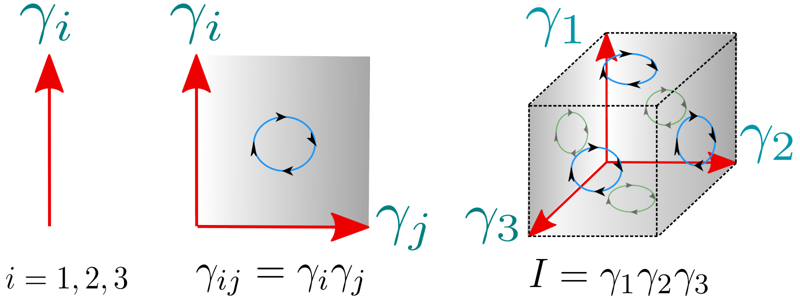

Figure 5 : The elements of $\mathcal{G}(\mathbb{R}^3)$ basis (besides the scalar 1): 3 vectors, 3 bivectors (oriented areas) γ_{ij}, and the trivector I (pseudoscalar/oriented volume).

Subalgebras and Isomorphism

- $\mathcal{G}^+(\mathbb{R}^2)$: Rotor Algebra of $\mathbb{R}^2$ has basis $\{1, \gamma_{12}\}$.

Subalgebras and Isomorphism

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- Isomorphic to the algebra of complex numbers.
  $A = c + \mathbf{j}d, \{c, d\} \in \mathbb{R} \rightarrow \mathbf{j}$ is a bivector!

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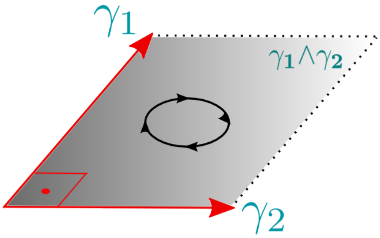

$$\{\gamma_1, \gamma_2\} \in \mathbb{R}^2$$
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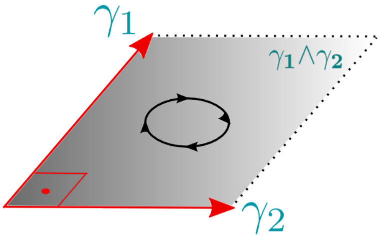

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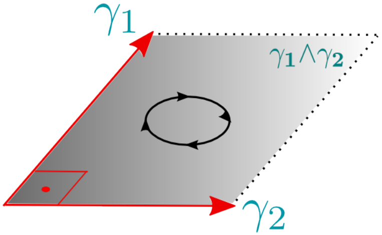

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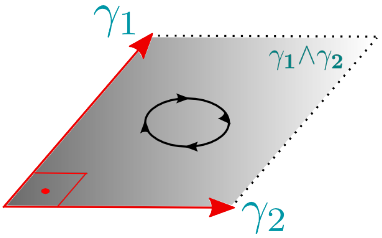

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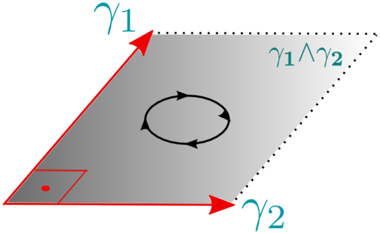

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- $\mathcal{G}^+(\mathbb{R}^3)$: Rotor Algebra of $\mathbb{R}^3$ has basis $\{1, \gamma_{12}, \gamma_{23}, \gamma_{31}\}$.

Subalgebras and Isomorphism

- $\mathcal{G}^+(\mathbb{R}^3)$: Rotor Algebra of $\mathbb{R}^3$ has basis $\{1, \gamma_{12}, \gamma_{23}, \gamma_{31}\}$.
- Isomorphic to the algebra of quaternions [2, 3]:

$$i \leftrightarrow -\gamma_{12} \qquad j \leftrightarrow -\gamma_{23} \qquad k \leftrightarrow -\gamma_{31}, \tag{5}$$

where $\{i, j, k\}$ are the three imaginary unities of quaternion algebra.

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  where $\{i, j, k\}$ are the three imaginary unities of quaternion algebra.
- Particularly useful in the development of GAAFs for pose estimation.
- Rotation operator:

$$x \to \underbrace{r x \widetilde{r}}_{rotated}, \tag{6}$$

  where $x \in \mathbb{R}^n$ and $r \in \mathcal{G}^+(\mathbb{R}^n)$.

Linear Estimation in GA

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Linear Estimation in GA

- Linear Algebra least-squares problem:

$$\min \left\| d - \hat{d} \right\|^2 \tag{7}$$

$\{d, \hat{d}\} \in \mathbb{R}^n, n = \{1, 2, \cdots\}$ and $\hat{d}$ is the estimate for d.

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$\{d, \hat{d}\} \in \mathbb{R}^n, n = \{1, 2, \cdots\}$ and $\hat{d}$ is the estimate for d.

- Two special cases:

  1. Standard: d is a multivector, $\hat{d} = u^* w$ is also a multivector resultant from an array product, u and w are $M \times 1$ arrays of multivectors
  2. Pose estimation: $\{d, x\} \in \mathbb{R}^n$ are vectors, $\hat{d} = r x \widetilde{r} + t$, where the rotor r and its reversed version $\widetilde{r}$ rotate x, and t is a $n \times 1$ translation vector

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General Cost Function in GA

GA general cost function

$$J(D, A_k, X, B_k) = \left| D - \sum_{k=1}^{M} A_k X B_k \right|^2$$

$\underbrace{\sum_{k=1}^{M} \widetilde{U}_k W_k}_{\text{array prod.}}$

$D = d \quad A_k = \widetilde{U}_k$
$M = 1 \quad B_k = W_k$
$X = 1$

$$J_s(w) = |d - u^* w|^2$$

GAAFs standard

$D = y \quad A_k = r$
$M = 1 \quad B_k = \widetilde{r}$
$X = x$

$$J_p(r) = |y - r x \widetilde{r}|^2$$
$$\text{subject to } r\widetilde{r} = \widetilde{r}r = 1$$

GAAFs pose estimation

- $\sum_{k=1}^{M} A_k X B_k \rightarrow$ canonical form of a linear transformation applied to the multivector X ([1, p.64 and p.121])

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Geometric-Algebra Adaptive Filters (Standard)

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Geometric-Algebra Adaptive Filters (Standard)

- Minimize

$$\boxed{J_s(w) = |d - u^* w|^2} \tag{8}$$

- Steepest-descent rule is adopted $\rightarrow$ follow the opposite direction of the gradient of the cost function

- Omitting the calculations here (please refer to Section 5.1 in the thesis)

GA-LMS (Standard)

- Update rule:

$$\boxed{w_i = w_{i-1} + \mu u_i e(i)}, \tag{9}$$

where μ is the step size and $e(i) = d(i) - \underbrace{u_i^* w_i}_{\text{array product}}$

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- Same shape of the regular LMS AFs

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- No constraints on the entries of the arrays u and w $\rightarrow$ they can be any kind of multivector

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- Same shape of the regular LMS AFs
- No constraints on the entries of the arrays u and w $\rightarrow$ they can be any kind of multivector
- It generalizes the standard LMS AF for several types of entries: general multivectors, rotors, quaternions, complex numbers, real numbers.

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Steady-State Analysis

- The performance analysis adopts an specific data model (see Section 5.2)

Steady-State Analysis

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- Starting point:

$$\boldsymbol{w}_i = \boldsymbol{w}_{i-1} + \mu \boldsymbol{u}_i f(\boldsymbol{e}(i)), \tag{10}$$

  where $f(\cdot)$ is a multivector-valued function of the estimation error $\boldsymbol{e}(i)$.

- **Boldface** $\rightarrow$ random quantity

Steady-State Analysis

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  where $f(\cdot)$ is a multivector-valued function of the estimation error $\boldsymbol{e}(i)$.

- **Boldface** $\rightarrow$ random quantity
- Energy Conservation Relations (ECR) [4]
- The ECR technique performs an interplay between the energies of the weight array $\boldsymbol{w}$ and the error $\boldsymbol{e}$ at two successive time instants:

$i-1$ (*a priori*) and i (*a posteriori*)

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Steady-State Analysis

- The variance relation is obtained

$$\boxed{2\mathsf{E}\left\langle \boldsymbol{e}_a(i)\widetilde{f}\right\rangle = \mu\mathsf{E}\left|\boldsymbol{u}_i f\right|^2}, \tag{11}$$

where $\boldsymbol{e}_a(i) = \boldsymbol{u}_i^*\Delta\boldsymbol{w}_{i-1}$ is the *a priori* error, in which $\Delta\boldsymbol{w}_i = w^o - \boldsymbol{w}_i$, and w^o is the optimal weight vector.

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- For the GA-LMS, $f(\boldsymbol{e}(i)) = \boldsymbol{e}(i) = \boldsymbol{e}_a(i) + \boldsymbol{v}(i)$.

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- Separation principle (see $[4, p.245]$)
- Analysis is valid for inputs drawn from a white Gaussian stochastic process

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- Separation principle (see $[4, p.245]$)
- Analysis is valid for inputs drawn from a white Gaussian stochastic process
- Steady-state EMSE for the complete algebra $\mathcal{G}(\mathbb{R}^n)$

$$\boxed{\zeta_{\text{LMS}} = \frac{\mu M 4^n \sigma_u^2 \sigma_v^2}{2 - \mu M 2^n \sigma_u^2}, \quad i \to \infty}. \tag{12}$$

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Any subalgebra g $\mathcal{G}^g(\mathbb{R}^n)$	$\dfrac{\mu M \binom{n}{g}^2 \sigma_u^2 \sigma_v^2}{2 - \mu M \binom{n}{g} \sigma_u^2}$
Even Algebras $\mathcal{G}^+(\mathbb{R}^n)$	$\dfrac{\mu M \left[\sum\limits_k \binom{n}{2k} \right]^2 \sigma_u^2 \sigma_v^2}{2 - \mu M \sigma_u^2 \sum\limits_k \binom{n}{2k}}$, for $k = 0, 1, 2, 3, \cdots$
Complete GA of $\mathbb{R}^3$ $\mathcal{G}(\mathbb{R}^3)$	$\dfrac{32 \mu M \sigma_u^2 \sigma_v^2}{1 - 4 \mu M \sigma_u^2}$
Rotor GA of $\mathbb{R}^3$ (Quaternions) $\mathcal{G}^+(\mathbb{R}^3)$	$\dfrac{\mu M \left[\binom{3}{0} + \binom{3}{2} \right]^2 \sigma_u^2 \sigma_v^2}{2 - \mu M \sigma_u^2 \left[\binom{3}{0} + \binom{3}{2} \right]}$
Rotor GA of $\mathbb{R}^2$ (Complex) $\mathcal{G}^+(\mathbb{R}^2)$	$\dfrac{\mu M \left[\binom{2}{0} + \binom{2}{2} \right]^2 \sigma_u^2 \sigma_v^2}{2 - \mu M \sigma_u^2 \left[\binom{2}{0} + \binom{2}{2} \right]}$
Rotor GA of $\mathbb{R}$ (Real) $\mathcal{G}^+(\mathbb{R})$	$\dfrac{\mu M \sigma_u^2 \sigma_v^2}{2 - \mu M \sigma_u^2}$

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Geometric-Algebra Adaptive Filters (Pose Estimation)

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Standard Rotation Estimation

- Two PCDs in the $\mathbb{R}^3$, Y (*Target*) and X (*Source*), with K correspondence points
- Match PCDs centroids

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Standard Rotation Estimation

- Two PCDs in the $\mathbb{R}^3$, Y (*Target*) and X (*Source*), with K correspondence points
- Match PCDs centroids
- Rigid transformation? *Least-squares problem*

$$\mathcal{F}(R) = \frac{1}{K}\sum_{k=1}^{K} \|y_k - Rx_k\|^2 \text{, subject to } R^*R = RR^* = I_d \text{ and } t = \bar{y} - R\bar{x} \tag{13}$$

R: 3×3 rotation matrix, t: 3×1 translation vector.

- Solution: SVD-based algorithms $\rightarrow$ *Outlier sensitive*

The Rotation Estimation Problem in GA

Recast (13) in GA:

$$e_k = y_k - Rx_k \Rightarrow e_k = y_k - \underbrace{r x_k \widetilde{r}}_{\text{Geometric Product}} \text{, subject to } r\widetilde{r} = \widetilde{r}r = |r|^2 = 1. \qquad (14)$$

The least-squares cost function becomes

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The Rotation Estimation Problem in GA

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The least-squares cost function becomes

$$J(r) = \frac{1}{K}\sum_{k=1}^{K} |y_k - r x_k \widetilde{r}|^2 = \frac{1}{K}\sum_{k=1}^{K} |e_k|^2 = \frac{1}{K}\sum_{k=1}^{K} e_k * \widetilde{e}_k = \frac{1}{K}\sum_{k=1}^{K} \langle e_k \widetilde{e}_k \rangle. \tag{15}$$

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GA-LMS (Pose Estimation)

- The *Geometric-algebra steepest-descent* algorithm for pose estimation

$$\boxed{r_i = r_{i-1} + \mu \frac{4}{m} \left[\sum_{k=1}^{m} y_k \wedge (r_{i-1} x_k \widetilde{r}_{i-1}) \right] r_{i-1}}, \tag{16}$$

- If $m = 1$ (one pair per iteration), $\widetilde{\nabla} J(r)$ is approximated by its *current value*

$$\frac{4}{K} \left[\sum_{k=1}^{K} y_k \wedge (r_{i-1} x_k \widetilde{r}_{i-1}) \right] r_{i-1} \approx 4 \left[y_i \wedge (r_{i-1} x_i \widetilde{r}_{i-1}) \right] r_{i-1}. \tag{17}$$

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GA-LMS (Pose Estimation)

- Update rule for the GA-LMS for pose estimation is

$$\boxed{r_i = r_{i-1} + \mu \left[y_i \wedge \left(r_{i-1} x_i \widetilde{r}_{i-1} \right) \right] r_{i-1}}, \tag{18}$$

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GA-LMS (Pose Estimation)

- Update rule for the GA-LMS for pose estimation is

$$\boxed{r_i = r_{i-1} + \mu \left[y_i \wedge \left(r_{i-1} x_i \widetilde{r}_{i-1} \right) \right] r_{i-1}}, \tag{18}$$

- *"Look at" one correspondence at each iteration*

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GA-LMS (Pose Estimation)

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- *"Look at" one correspondence at each iteration*
- *Enforces reduction in computational complexity* compared to traditional rotation estimation techniques
  - GA-LMS cost: 54 real multiplications and 39 real additions per iteration.
  - SVD-based methods have cost $O(K)$ at each registration iteration, i.e., it depends on the number of points in the PCDs.

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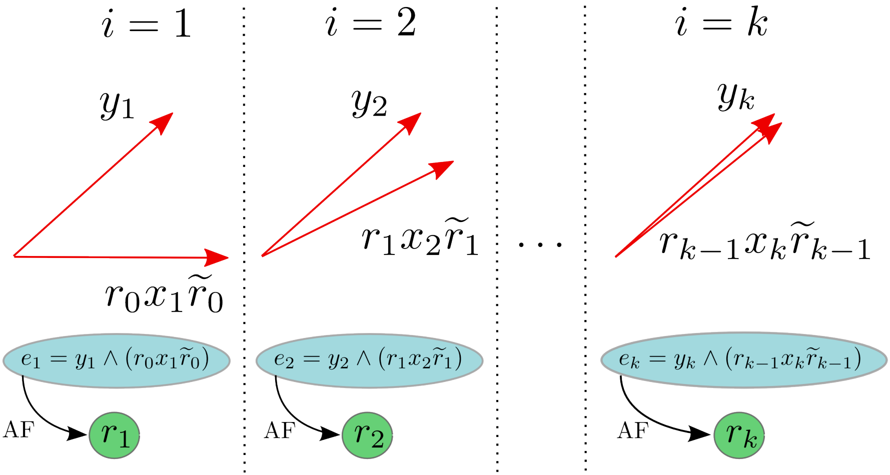

Figure 6 : Step-by-step GA-LMS (pose estimation)

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Step-Size Bounds

$$\boxed{\mu(Y,X) = \rho \sum_{k=1}^{K} \langle y_k x_k Q \rangle \Big/ \sum_{k=1}^{K} \langle y_k \widetilde{Q} x_k Q \rangle}, \quad Q = \sum_{n=1}^{K} y_n \wedge x_n. \tag{19}$$

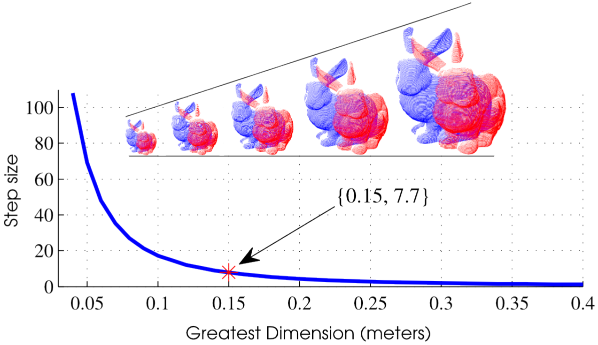

Figure 7 : Simple rule for selecting μ for the Stanford Bunny set. $\rho = 15$.

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Implementation in C++

- The computational implementation of outer and geometric products requires special libraries and/or toolboxes
- *Geometric Algebra ALgorithms Expression Templates* (GAALET) [5], a C++ library for evaluation of GA expressions
- Codes and scripts on openga.org

openga.org

docker

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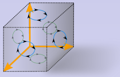
System Identification with Standard GAAFs

- $\mathcal{G}(\mathbb{R}^3)$ Multivector Entries

$$w^o = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_M \end{bmatrix} = \begin{bmatrix} 0.55 + 0\gamma_1 + 1\gamma_2 + 2\gamma_3 + 0.71\gamma_{12} + 1.3\gamma_{23} + 4.5\gamma_{31} + 3I \\ 0.55 + 0\gamma_1 + 1\gamma_2 + 2\gamma_3 + 0.71\gamma_{12} + 1.3\gamma_{23} + 4.5\gamma_{31} + 3I \\ \vdots \\ 0.55 + 0\gamma_1 + 1\gamma_2 + 2\gamma_3 + 0.71\gamma_{12} + 1.3\gamma_{23} + 4.5\gamma_{31} + 3I \end{bmatrix} \quad (20)$$



System Identification with Standard GAAFs

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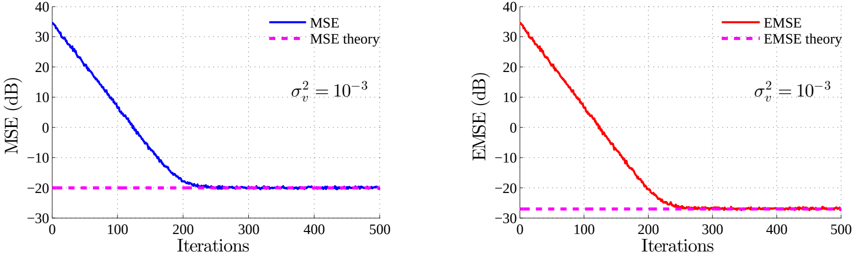

Figure 8 : GA-LMS: MSE and EMSE learning curves for $M = 10$, $\mu = 0.005$, and $\sigma_v^2 = 10^{-3}$. The curves are averaged over 100 experiments.

GAAFs

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System Identification with Standard GAAFs

- $\mathcal{G}(\mathbb{R}^3)$ Multivector Entries
- $\sigma_v^2 = \{10^{-2}, 10^{-3}, 10^{-5}\}$.
- The simulated steady-state value is obtained by averaging the last 200 points of the ensemble-average learning curve for each M.

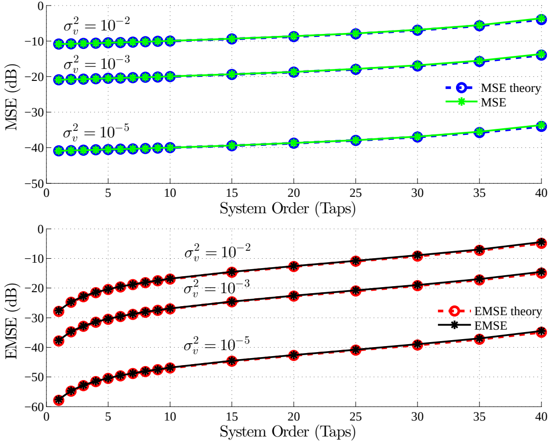

Figure 9 : Steady-state MSE and EMSE as functions of the system order M

System Identification with Standard GAAFs

- $\mathcal{G}^+(\mathbb{R}^3)$ Rotor Entries

- (a) EMSE, $M = 10$, $\mu = 0.005$, and $\sigma_v^2 = 10^{-3}$ (100 experiments).
- (b) MSE and EMSE versus the number of taps for $\mu = 0.005$ and $\sigma_v^2 = 10^{-3}$.

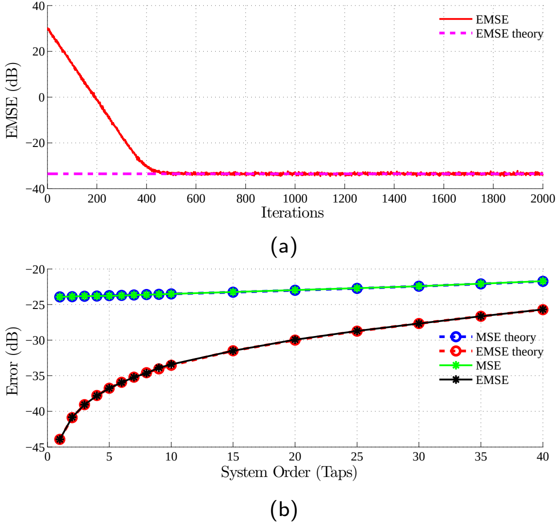

Figure 10 : Rotor entries.

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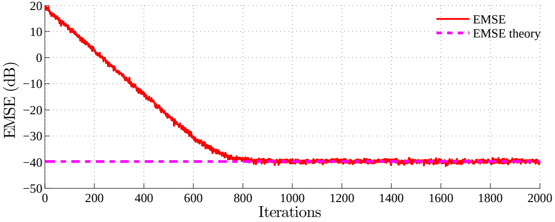
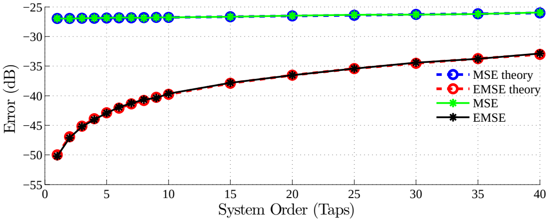
- $\mathcal{G}^+(\mathbb{R}^2)$ Complex Entries

- (a) EMSE learning curve for $M = 10$, $\mu = 0.005$, and $\sigma_v^2 = 10^{-3}$ (100 experiments).
- (b) Steady-state MSE and EMSE versus the number of taps for $\mu = 0.005$ and $\sigma_v^2 = 10^{-3}$.



(a)



(b)

Figure 11 : Complex entries.

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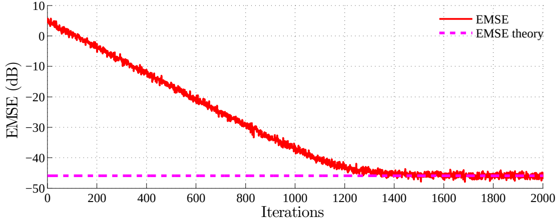
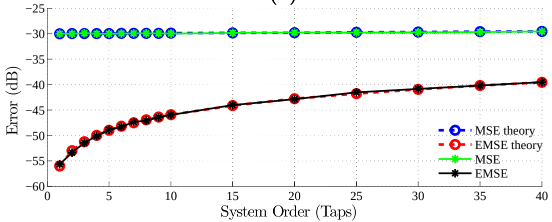
System Identification with Standard GAAFs

- $\mathcal{G}^+(\mathbb{R})$ Real Entries

- (a) EMSE learning curve for $M = 10$, $\mu = 0.005$, and $\sigma_v^2 = 10^{-3}$ (100 experiments).
- (b) Steady-state MSE and EMSE versus the number of taps for $\mu = 0.005$ and $\sigma_v^2 = 10^{-3}$.



(a)



(b)

Figure 12 : Real entries.

3D Registration of Point Clouds with GAAFs for Pose Estimation

- VIDEO
- Cube set.
- (top) EMSE for $\sigma_v^2 = 10^{-5}$ and different values of μ.
- (bottom) EMSE for $\mu = 0.2$ and different noise variances σ_v^2. Averaged over 200 realizations.

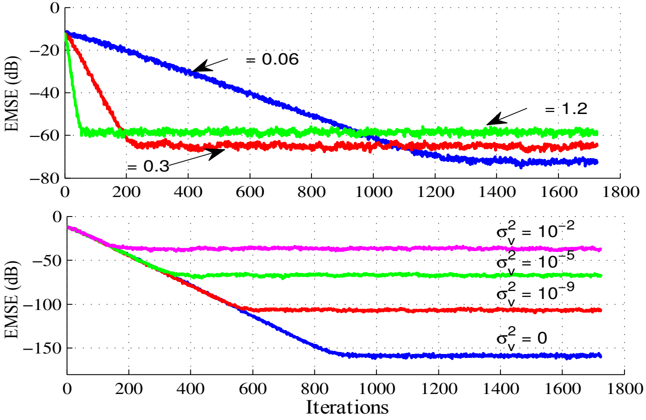


Figure 13 : Cube set.

- Stanford bunnies set.
- The cost function curve is plotted on top of the MSE to emphasize the minimization performed by the AF.

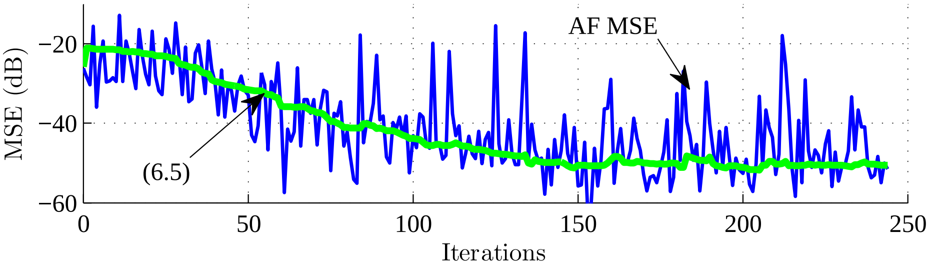


Figure 14 : Bunny set, $\mu = 8$.

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- The majority of AF algorithms available in the literature resorts to specific subalgebras of GA (real, complex numbers and quaternions).

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- The development of the GAAFs is an attempt to unify those different AF approaches under the same mathematical language.

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Conclusion

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- GAAFs have improved estimation capabilities $\rightarrow$ they can naturally estimate any kind of multivector.

Conclusion

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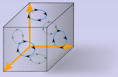
- GAAFs have improved estimation capabilities $\rightarrow$ they can naturally estimate any kind of multivector.
- Update rule shape is invariant (GAAFs standard) with respect to the multivector subalgebra.
- It is expected that any estimation problem posed in terms of hyper-complex quantities could benefit from this work.



Conclusion

- The majority of AF algorithms available in the literature resorts to specific subalgebras of GA (real, complex numbers and quaternions).
- The development of the GAAFs is an attempt to unify those different AF approaches under the same mathematical language.
- GAAFs have improved estimation capabilities → they can naturally estimate any kind of multivector.
- Update rule shape is invariant (GAAFs standard) with respect to the multivector subalgebra.
- It is expected that any estimation problem posed in terms of hyper-complex quantities could benefit from this work.
- openga.org → a hub for GA-based algorithms.

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Papers

1. *"Geometric-algebra LMS adaptive filter and its application to rotation estimation"*- Published in IEEE Signal Processing Letters 2016.
2. *"6DOF point-cloud alignment using geometric algebra-based adaptive filtering"*- Proceedings of IEEE Winter Conference on Applications of Computer Vision (WACV) 2016.
3. *"Analyzing LIDAR Scan Skewing and its Impact on Scan Matching"*- IEEE conference on Indoor Positioning and Indoor Navigation (IPIN) 2016 (paper accepted).
4. *"Geometric-Algebra Adaptive Filters"*- to be submitted to IEEE Transactions on Signal Processing.

Future Work

- GA-NLMS and GA-RLS for system ID.
- GA-NLMS and GA-RLS for pose estimation.
- Mean-square analysis of the GAAFs for pose estimation.

References I

D. Hestenes and G. Sobczyk,
Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics,
Fundamental Theories of Physics. Springer Netherlands, 1987.

E. Hitzer,
"Introduction to Clifford's Geometric Algebra,"
Journal of the Society of Instrument and Control Engineers, vol. 51, no. 4, pp. 338–350, 2012.

Erik B. Dam, Martin Koch, and Martin Lillholm,
"Quaternions, interpolation and animation - diku-tr-98/5,"
Tech. Rep., Department of Computer Science, University of Copenhagen, 1998. Available: http://web.mit.edu/2.998/www/QuaternionReport1.pdf.

A.H. Sayed,
Adaptive filters,
Wiley-IEEE Press, 2008.

References II

Florian Seybold and U Wössner,
“Gaalet-a c++ expression template library for implementing geometric algebra,”
in *6th High-End Visualization Workshop*, 2010.

Thank you!

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